Hit problems and the Steenrod algebra
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This course of lectures will make intelligible a number of problems, listed in the last section, concerning the action of the Steenrod algebra on polynomials. The subject matter is rather technical but we shall try to indicate how some of the problems fit into the wider context of algebraic topology and invariant theory.

Contents

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The first section introduces the hit problem in a general algebraic setting for graded left modules over a graded ring with a right semigroup action. A couple of examples from topology and invariant theory illustrate the ideas. Then we restrict attention to the main example which is to do with the Steenrod algebra $\mathcal{A}$ at the prime 2 acting on the polynomial algebra $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ in n variables x_i over the field $\mathbb{F}_2$ of two elements, with the right action of the matrix semigroup $M(n, \mathbb{F}_2)$.

The next section describes recent joint work with Ali Janfada on hit problems for symmetric polynomials.

In the third section we explain some interconnections between modular representation theory of the semigroup algebra $\mathbb{F}_2[M(n, \mathbb{F}_2]$ and the splitting theory of the stable types of the classifying spaces of certain groups. We describe some recent work with Grant Walker on the Steinberg representation and more general questions about the linkage of first occurrences of irreducible representations via Steenrod operations in the polynomial algebra.

In the fourth section the scope of the investigation is extended to the differential operator algebra $\mathcal{D}$, as the ring of operators, which is the natural setting for studying hit problems over the integers and at odd primes. We refer to [49] for a fairly extensive bibliography concerning the action of the Steenrod algebra on polynomials.

1 The hit problem

For any graded left module $\mathbf{M}$ over a graded ring $\mathcal{R}$ with unit, we write $\mathbf{M}^d$ for the elements of grading d. An element $f \in \mathbf{M}^d$ is called *hit* if it can be written as a finite sum

$$f = \sum_i \Theta_i f_i,$$

where the elements Θ_i belong to $\mathcal{R}$ and the f_i are homogeneous elements in $\mathbf{M}$ of grading strictly less than d. We refer to this representation of f as a *hit equation.* The hit elements form a submodule $\mathbf{N}$ of $\mathbf{M}$. The quotient $\mathbf{Q} = \mathbf{Q}(\mathbf{M}) = \mathbf{M}/\mathbf{N}$ is essentially a graded abelian group because the ring acts trivially. A minimal generating set for $\mathbf{Q}$ lifts to a minimal generating set for $\mathbf{M}$ as a module over $\mathcal{R}$. For the sake of brevity we shall sometimes say that f is equivalent to g in $\mathbf{Q}$ and write $f \cong g$ when, strictly speaking, we mean $f - g$ is hit in $\mathbf{M}$ and the equivalence classes of f and g are equal in $\mathbf{Q}$.

In particular we can view $\mathbf{R} = \mathcal{R}$ as left module over itself. In this case a hit element is traditionally called *decomposable* and the hit problem is then concerned with writing an element of $\mathbf{R}$ as a sum of products of elements of lower grading.

For present purposes we concentrate on the restricted situation where our modules are vector spaces over a field $\mathbb{F}$, have no elements of negative grading and are of finite type, which means that $\dim(\mathbf{M}^d)$ is finite for each d. We also assume that $\mathcal{R}$ is a connected algebra over $\mathbb{F}$, which means that $\mathcal{R}_0 = \mathbb{F}$. Topological motivation for studying these objects is provided by the example of the cohomology $\mathbf{M} = \mathbf{H}^*(X; \mathbb{F}_2)$ of a complex X of finite type over $\mathbb{F}_2$ under the left action of the Steenrod algebra $\mathcal{A}$.

It is worth pointing out here how decomposability can sometimes be the last stage in an argument requiring several intermediate steps of a geometric or topological nature. A famous example is the solution by Frank Adams of the problem about non-singular multiplications on Euclidean space. Early pioneers in topology translated the problem through geometry and topology into a question about the cohomology ring of a certain topological space under the action of the Steenrod algebra. Without going into details at this stage, we note that $\mathcal{A}$ is generated by elements Sq^r called *Steenrod squares* in gradings $r \geq 0$ subject to certain relations. It turns out that all Steenrod squares in positive grading are decomposable in $\mathcal{A}$ except when $r = 2^k$ for some k. It was this fact which first led Adem to a proof that non-singular multiplications on Euclidean space $\mathbf{R}^n$ cannot exist for dimensions other than $n = 2^k$. By extending the notion of decomposability into the broader context of 'secondary' operations Adams succeded in decomposing Sq^{2^k} for $k > 3$, thereby proving the long oustanding conjecture that non-singular multiplications on Euclidean space can only exist for $n = 1, 2, 4, 8$, where they are realised by real, complex, quaternionic and Cayley multiplication.

We shall be concerned in this course with a number of related questions.

Problems 1.1 *1. Find a criterion for $f \in \mathbf{M}^d$ to be hit.*

2. Find a minimal generating set for $\mathbf{M}$ or, equivalently, a basis of $\mathbf{Q}$.

3. When is the dimension of $\mathbf{Q}^d$ equal to zero?

4. Is the dimension of $\mathbf{Q}^d$ bounded independently of d?

The hit problem can be enhanced by introducing a right action of a group or semigroup Γ on the module $\mathbf{M}$ compatible with the left action of $\mathcal{R}$. To be precise, we suppose that each $\mathbf{M}^d$ is a right representation of Γ and for each $\Theta \in \mathcal{R}^t$ the left linear map $\Theta\colon \mathbf{M}^d \to \mathbf{M}^{d+t}$ is a map of right Γ-modules. For $\Theta \in \mathcal{R}, x \in \mathbf{M}, \pi \in \Gamma$ we can write $\Theta x \pi$ unambiguously. Hit problems then receive an equivariant flavour. For example, the quotient $\mathbf{Q}$ becomes a graded representation of Γ. We can also study hit problems for the fixed point set $\mathbf{M}^\Gamma$ as an $\mathcal{R}$-submodule of $\mathbf{M}$. More generally, we can examine the decomposition of $\mathbf{M}$ into summands afforded by idempotents in the semigroup algebra of Γ. Each such summand is then an $\mathcal{R}$-submodule of $\mathbf{M}$ and the investigation of hit problems for these summands is intimately related to the modular representation theory of Γ over the ground field $\mathbb{F}$. In the case of the Steenrod algebra and the matrix semigroup, there are also topological implications to do with the stable splitting of classifying spaces.

So we study the case where $\Gamma = M(n, \mathbb{F}_2)$ is the semigroup of $n \times n$ matrices $A = (a_{ij})$ over $\mathbb{F}_2$ acting on the right of $\mathbf{P}(n)$ by linear substitution of variables,

$$x_i A = \sum_s a_{is} x_s.$$

In this case, Γ contains the subgroup of non-singular matrices $GL(n, \mathbb{F}_2)$, which in turn contains the symmetric group Σ_n consisting of matrices with a single non-zero entry in each row or column. The action extends to the 'rook' semigroup of all matrices over $\mathbb{F}_2$ with at most one non-zero entry in each row or column. Let $\mathbf{B}(n) = \mathbf{P}(n)^{\Sigma_n}$ be the ring of invariants, in other words the symmetric polynomials in $\mathbf{P}(n)$. It turns out that the Steenrod squares commute with the action of $M(n, \mathbb{F}_2)$, in particular with Σ_n and $GL(n, \mathbb{F}_2)$. This raises some interesting hit problems in $\mathbf{B}(n)$, and in the *Dickson algebra* $\mathbf{D}(n) = \mathbf{P}(n)^{GL(n, \mathbb{F}_2)}$.

Before embarking on the main topic, we consider a hit problem in which the ring of invariants plays a different role, this time as the ring of operators $\mathcal{R}$ rather than the module acted on. The example is taken from Larry Smith's book on invariant theory [37], recouched in the language of an equivariant hit problem.

Example 1.2 *Let $\mathbf{M} = \mathbb{Q}[x_1, \cdots, x_n]$ be the polynomial algebra over the rationals in n variables. Let Σ_n act on the right of $\mathbf{M}$ in the usual way. Take for $\mathcal{R}$ the ring of symmetric polynomials in $\mathbf{M}$ acting on the left of $\mathbf{M}$ by the usual multiplication of polynomials. Clearly the $\mathcal{R}$-action commutes with the Σ_n action.*

For this example we can answer some of the questions posed in Problems 1.1.

1. All homogeneous polynomials of degree greater than $n(n-1)/2$ are hit.
2. A basis for $\mathbf{Q}$ consists of the $n!$ monomials $x_1^{i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}$, where $0 \le i_r \le r$.
3. In fact $\mathbf{Q}$ is a graded version of the regular representation of Σ_n.

For example, in the case $n=3$, the monomials

$$1, x_1, x_2, x_1^2, x_2^2, x_1^2x_2$$

generate $\mathbf{Q}$. From elementary representation theory, it is known that the three irreducible representations of Σ_3 over the rationals must appear in $\mathbf{Q}$ with multiplicity equal to their dimension. Indeed, the trivial representation appears once, generated by 1 in grading 0. The sign representation appears once, generated by $x_1^2x_2$ in grading 3, and the irreducible 2-dimensional representation of Σ_3 appears twice, generated by x_1, x_2 in grading 1 and by x_1^2, x_2^2 in grading 2. Every homogeneous polynomial f of degree at least 4 is hit; in other words, it can be written in the form

$$f = \Theta_1 + \Theta_2 x_1 + \Theta_3 x_2 + \Theta_4 x_1^2 + \Theta_5 x_2^2 + \Theta_6 x_1^2 x_2,$$

where the Θ_i are symmetric polynomials of positive degree.

This example has some special features which will not apply in general. For example, the product of a hit element by a polynomial is also hit because the ring of operators commutes with multiplication of polynomials. Hence the hit elements form an ideal and $\mathbf{Q}$ is just the algebra of coinvariants [37] of the symmetric group. We shall refer back to this example in the last section.

The algebras $\mathbf{P}(n)$ and subalgebra $\mathbf{B}(n)$ of $\mathbf{P}(n)$ are of particular interest in topology because they realise respectively the cohomology of the product of n copies of infinite real projective space and the cohomology of the classifying space $BO(n)$ of the orthogonal group $O(n)$. This is the universal place for studying Stiefel-Whitney classes of manifolds. Symmetric polynomials, divisible by the product of the variables $x_1 \cdots x_n$ also has a topological interpretation as the cohomology $\mathbf{M}(n) = H^*(MO(n), \mathbb{F}_2)$ of the Thom space $MO(n)$ in positive dimensions. Thom spaces are important in studying the immersion and embedding theory of manifolds.

1.1 The action of the Steenrod algebra on polynomials

In this section we explain how Steenrod squares act on polynomials and state some facts about the hit problem for $\mathbf{P}(n)$.

N.B. Throughout these lectures we adopt the non-standard convention of writing numbers in *reversed* dyadic expansion. For example 0101 is the reversed dyadic expansion of the number 10. Dyadic positions are counted from 0 on the left. It is customary to denote by $\alpha(d)$ the number of digits 1 in the expansion of d. We call this the α-count of d. For future reference it should be noted that if d has a digit 1 in position k then $\alpha(d+2^k) \leq \alpha(d)$ and there is strict inequality if d also has a digit 1 in position $k+1$ because of the carry forward effect of binary addition.

Another numerical function that features frequently in this subject is $\mu(d)$, which is the least number k for which it is possible to write $d = \sum_{i=1}^{k}(2^{\epsilon_i} - 1)$.

The Steenrod algebra $\mathcal{A}$ is defined to be the graded algebra over the field $\mathbb{F}_2$, generated by symbols Sq^k, called Steenrod squares, in grading k, for $k \geq 0$, subject to the Adem relations [40] and $Sq^0 = 1$. For present purposes we need to know that the Steenrod algebra acts by composition of linear operators on $\mathbf{P}(n)$ and the action of the Steenrod squares $Sq^k\colon \mathbf{P}^d(n) \to \mathbf{P}^{d+k}(n)$ on monomials $f, g \in \mathbf{P}(n)$ is determined by the following rules [49].

Proposition 1.3 *1. $Sq^k f = f^2$ if $\deg(f) = k$ and $Sq^k f = 0$ if $\deg(f) < k$.*

2. The Cartan formula $Sq^k(fg) = \sum_{0 \leq r \leq k} Sq^r(f)Sq^{k-r}(g)$.

In principle these rules enable the evaluation of a Steenrod operation on any polynomial by induction on degree.

The next results are elementary consequences of the rules in Proposition 1.3.

Proposition 1.4 *Let $f = x_1^{d_1} \cdots x_n^{d_n}$ be a monomial in $\mathbf{P}(n)$ and x a typical variable.*

1. *$Sq^r(f) = \sum_{r_1+\cdots+r_n=r} Sq^{r_1}(x_1^{d_1}) \cdots Sq^{r_n}(x_n^{d_n})$.*

2. *$Sq^r(x^d) = 0$ unless, for each position j where there is a binary digit 1 of r, there is also a binary digit 1 of d in position j. In this case $Sq^r(x^d) = x^{r+d}$. In particular, $Sq^{2^k}(x^d) = x^{2^k+d}$ if and only if d has a digit 1 in position k.*

3. *The power x^r is in the image of a positive Steenrod square if and only if r is not of the form $2^\epsilon - 1$.*

4. *If r is odd then $Sq^r(f^2) = 0$, whereas $Sq^{2r}(f^2) = (Sq^r(f))^2$. Consequently the action of the Steenrod algebra on $\mathbf{P}(n)$ is 'fractal' in the sense that a copy of the algebra acts on squares of polynomials by duplication of the suffices of the operators.*

5. *Steenrod squares commute with the right action of the symmetric group Σ_n, which permutes the variables $x_1, \ldots, x_n$.*

6. *Steenrod squares commute with the right action of the full semigroup of $n \times n$ matrices acting by linear substitution in the variables [49].*

Item 5 explains algebraically why $\mathbf{B}(n)$ is a submodule of $\mathbf{P}(n)$. Item 6 is a stronger statement and explains why the Dickson algebra $\mathbf{D}(n)$ is a module over $\mathcal{A}$. There are topological reasons why $\mathbf{B}(n)$ and $\mathbf{M}(n)$ are modules over the Steenrod algebra because these are cohomology algebras of certain topological spaces. For $\mathbf{D}(n)$, however, this is not the case if $n > 5$ [38]. So invariance of $\mathbf{D}(n)$ under $\mathcal{A}$ is an algebraic bonus. We shall say more about this matter in section 3.

We recall [49] that a monomial $x_1^{d_1} \cdots x_n^{d_n}$ is called a *spike* if every exponent d_i is of the form $2^{\epsilon_i} - 1$. It follows from items 1, 3 of Proposition 1.4 that a spike can never appear as a term in a hit polynomial in $\mathbf{P}(n)$ when written irredundantly. Hence the spikes must always appear in any set of minimal generators of the module $\mathbf{P}(n)$.

There are a few deeper facts about the Steenrod algebra which are needed later to analyse the hit problem. A string of Steenrod squares

$$Sq^{k_1} Sq^{k_2} \cdots Sq^{k_t}$$

of length $t \geq 1$. is called *admissible* if $k_i \geq 2k_{i+1}$ for $1 \leq i < t$. This includes all Sq^i as admissible for $i \geq 0$.

Proposition 1.5 *As a vector space, $\mathcal{A}$ is generated by the admissible strings of Steenrod squares.*

Also important for hit problems is the following result already referred to earlier.

Proposition 1.6 *As an algebra, $\mathcal{A}$ is generated by the Steenrod squares Sq^{2^k} for $k \geq 0$.*

In the first place a hit equation for $f \in \mathbf{P}(n)$ has the general form $f = \sum_i \Theta_i f_i$, where the elements $\Theta_i \in \mathcal{A}$ have positive grading, but because the Sq^i generate $\mathcal{A}$ there must then be a hit equation of the form $f = \sum_{i>0} Sq^i g_i$ and, in the light of Proposition 1.6, f will also satisfy a hit equation of the form

$$f = \sum_{k \geq 0} Sq^{2^k} h_k,$$

where f_i, g_i, h_k are homogeneous elements of $\mathbf{P}(n)$.

The Steenrod algebra is a Hopf algebra with diagonal defined by

$$\psi(Sq^k) = \sum_{0 \leq i \leq k} Sq^i \otimes Sq^{k-i}.$$

It then admits a *conjugation* operator χ, which is an anti-automorphism of order 2. For an element $\Theta \in \mathcal{A}$ we use the notation $\chi(\Theta) = \widehat{\Theta}$. Conjugation satisfies the recursion formulae

$$\sum_{i=0}^{k} Sq^i \widehat{Sq^{k-i}} = 0,$$

for $k > 0$, from which it is possible, at least in principle, to work out $\widehat{Sq^k}$ in terms of Steenrod squares by induction on k.

The following formulae are useful for handling conjugate squaring operators.

Proposition 1.7 *There is a conjugate Cartan formula*

$$\widehat{Sq^k}(fg) = \sum_{0 \le r \le k} \widehat{Sq^r}(f)\widehat{Sq^{k-r}}(g)$$

and evaluation of a conjugate square on powers of a single variable is given by

$$\widehat{Sq^r}(x^{2^k}) = \begin{cases} x^{2^m}, & \text{if } r = 2^m - 2^k,\ m \ge k, \\ 0, & \text{otherwise.} \end{cases}$$

As with Steenrod squares themselves, these formulae enable the evaulation of a conjugate operation on any polynomial by induction on degree.

The next result plays a major role in solving hit problems [12, 13, 49].

Proposition 1.8 *Let u, v denote homogeneous elements in $\mathbf{P}(n)$. Then*

$$uSq^k(v) - \widehat{Sq^k}(u)v = \sum_{i>0} Sq^i(u\widehat{Sq^{k-i}}(v)).$$

An immediate consequence, known as the χ-trick, is that $uSq^k(v)$ is hit in $\mathbf{P}(n)$ if and only if $\widehat{Sq^k}(u)v$ is hit in $\mathbf{P}(n)$. By iterating the formula of Proposition 1.8 on compositions of Steenrod squares and using linearity, we obtain a more general statement.

Proposition 1.9 *Let u, v denote homogeneous elements in $\mathbf{P}(n)$. Then, for any $\Theta \in \mathcal{A}$, there is a hit equation of the form*

$$u\Theta(v) - \widehat{\Theta}(u)v = \sum_{i>0} Sq^i(\sum_{l} \Theta_{il}(u)\Phi_{il}(v)),$$

for certain elements $\Theta_{il}, \Phi_{il} \in \mathcal{A}$. In particular we have the equivalence $u\Theta(v) \cong \widehat{\Theta}(u)v$ in $\mathbf{Q}(\mathbf{P}(n))$.

The *excess* of an element Θ in $\mathcal{A}$ is defined as the smallest positive integer s such that $\Theta(x_1x_2\cdots x_s) \neq 0$. The following result goes back to Milnor [30, 35, 49].

Proposition 1.10 *The excess of $\widehat{Sq^k}$ is $\mu(k)$.*

This result has been improved in [26, 35].

Theorem 1.11 *The excess of $\widehat{Sq^d}\widehat{Sq^{2d}}\cdots\widehat{Sq^{2^{k-2}d}}\widehat{Sq^{2^{k-1}d}}$ is $(2^k - 1)\mu(d)$.*

From 1.8 and 1.11 we obtain the following corollary [35, 49].

Theorem 1.12 *Let* $f = uv^{2^k}$ *be a monomial in* $\mathbf{P}(n)$ *and suppose* $\deg(u) < (2^k - 1)\mu(\deg(v))$. *Then* f *is hit.*

To prove this statement, we write

$$\Theta = Sq^{2^{k-1}d} Sq^{2^{k-2}d} \cdots Sq^{2d} Sq^{d}$$

and observe that $v^{2^k} = \Theta(v)$ where $d = \deg(v)$. The χ-trick of Proposition 1.9 and Theorem 1.11 then complete the argument.

1.2 Binary blocks and order relations

As an intuitive aid to understanding many of the processes involving the action of Steenrod operations in $\mathbf{P}(n)$ it is useful to exhibit a monomial f as a binary block of digits 0 or 1 [8]. This means the matrix whose rows are the reversed binary expansions of the exponents of the variables $x_1, \ldots, x_n$ in f. We shall adopt the convention of denoting a monomial by a lower case letter and its binary block by the corresponding upper case letter. For example, the monomial $f = x_1^3 x_2^2 x_3^5$ is represented by the binary block

$$F = \begin{matrix} 1 & 1 & \\ 0 & 1 & \\ 1 & 0 & 1 \end{matrix}$$

Normal matrix notation will be used, except that the columns are counted from 0 to be consistent with 2-adic exponents. It should be noted in particular that the juxtaposition of two blocks UV corresponds to the monomial uv^{2^k}, where $k-1$ is the position of the last column of U. The double suffix notation $F_{(i,k)}$ refers to the entry of the binary block F in row i and column k. It is the digit in position k of the reversed binary expansion of d_i in the monomial $f = x_1^{d_1} \cdots x_n^{d_n}$. We shall occasionally use the notation $F_{(i)}$ to refer to row i of F.

There are several ways of ordering monomials of $\mathbf{P}(n)$ to be compatible with the action of the Steenrod algebra. The order relation used in [8, 49], called the ω-order, is defined as follows. Let $\omega_j(F) = \sum_i F_{(i,j)}$ denote the sum of the digits in column j of the binary block F. Now form the ω-vector $\omega(F) = (\omega_0(F), \omega_1(F), \ldots, \omega_k(F), \ldots)$ and order such vectors in left lexicographic order.

The transpose of the ω-order, which we shall call the α-order, is defined as follows. For a block F the α-counts of the rows of F are arranged as the components of a vector in non-decreasing order of magnitude from left to right. Such α-vectors are then compared lexicographically. This process defines the α-order relation on monomials and is again symmetric in the variables. For example, if the smallest α-count of the exponents in the monomial f is less than the smallest α-count of the exponents in the monomial g then $f <_\alpha g$. If these numbers are equal we look at the next smallest and so on.

The following statement explains the compatibility of the action of the Steenrod algebra with the order relations and is an easy consequence of items in Proposition 1.4.

Proposition 1.13 *Any monomial produced by the action of any positive element in the Steenrod algebra on a monomial f has strictly lower ω-order than that of f and no greater α-order.*

We shall say that a monomial f is ω-reducible if there is a hit equation of the form

$$f - g = \sum_i \Theta_i f_i,$$

for positively graded elements $\Theta_i \in \mathcal{A}$, where the ω-order of every monomial in g is lower than the ω-order of f. There is a similar definition for the α-order relation.

In the next couple of propositions the χ-trick is used to show how in certain situations the ω-order of a monomial can be reduced. Statements are sometimes more transparent when phrased in the language of binary blocks.

Proposition 1.14 *Let $F = UV$ be a binary block corresponding to a monomial $f = uv^{2^k}$ in $\mathbf{P}(n)$, where $k-1$ is the last column position of U. Suppose that v is hit. Then F is ω-reducible in $\mathbf{P}(n)$. In fact the ω-vector of the reduction can be assumed to be reduced in some position prior to k.*

To prove this result formally we first write

$$v = \sum_i \Theta_i(f_i)$$

for elements Θ_i of positive grading in the Steerod algebra and polynomials f_i in $\mathbf{P}(n)$. We then appeal to the fractal nature of the Steenrod algebra as explained in item 4 of Proposition 1.4 to write

$$v^{2^k} = \sum_i \Phi_i(f_i^{2^k}),$$

where the Φ_i are constructed from the Θ_i by iterated duplication of suffices in compositions of squaring operations. Then by the χ-trick of Proposition 1.9 we obtain the equivalence

$$uv^{2^k} = \sum_i u\Phi_i(f_i^{2^k}) \cong \sum_i (\widehat{\Phi}_i u) f_i^{2^k}.$$

Finally we apply Proposition 1.13 to see that the the ω-order of every monomial in $\widehat{\Phi}_i u$ is lower that the ω-order of u, indeed in a position prior to k. It follows that all monomials in $(\widehat{\Phi}_i u) f_i^{2^k}$ have ω-order lower than f as required.

We shall sometimes paraphrase proofs of this kind by saying simply that $F = UV$ is reducible because V is hit.

The following result is an immediate corollary.

Proposition 1.15 *Let F be a binary block with a zero column in position k, but with a non-zero column in some higher position. Then F is ω-reducible. Furthermore the reduction may be taken with no higher α-order than that of F.*

To demonstrate this result, suppose that the first zero column of F occurs at position k. Then split the block $F = UV$ vertically between positions $k-1$ and k, as in the previous proposition, where now the first column of V is zero. Then V is a perfect square and therefore hit. Proposition 1.14 completes the argument as far as reduction is concerned. The hit equation for V effectively moves all columns of V back one place, which does not change the α-count of the rows of V. An appeal to Proposition 1.13 finishes the proof.

1.3 Results for $\mathbf{P}(n)$

We shall now list a number of results about the hit problem for $\mathbf{P}(n)$ in answer to some of the questions posed in Problems 1.1. Most of these results are well known and can be found in various sources [1, 2, 8, 11, 22, 23, 31, 32, 34, 36, 44, 45, 46, 47, 49]. It should be mentioned a this point that we are taking the 'cohomology' approach to the hit problem. For the alternative 'homology' approach, we refer to [1, 11], where the problem is treated in terms of kernels of the adjoint action of Steenrod squares.

The solution of the Peterson conjecture [46, 47, 49] gives the following answer to question 3 of Problems 1.1.

Theorem 1.16 *The quotient space $\mathbf{Q}^d(\mathbf{P}(n))$ is zero if and only if $\mu(d) > n$.*

To prove this result we note first of all that in degree d, when $\mu(d) \leq n$, there is a spike which shows that the dimension of $\mathbf{Q}^d(\mathbf{P}(n))$ is non-zero. In the other direction, when $\mu(d) > n$, consider splitting a block $F = UV$ between column positions 0 and 1. Then $d = \deg U + 2 \deg V$. It can be seen that $\mu(\deg(V)) > \deg(U)$, otherwise we would be able to write

$$\deg(V) = \sum_{i=1}^{\deg(U)} (2^{\epsilon_i} - 1),$$

in which case

$$\deg U + 2 \deg V = \sum_{i=1}^{\deg(U)} (2^{\epsilon_i + 1} - 1),$$

contradicting $\mu(d) > n$, since $\deg U \leq n$. The result now follows from Theorem 1.12 in the case $k = 1$.

An answer to the second and fourth question in 1.1 can be found in in [22, 23, 8] for low values of n. The first three questions in 1.1 are answered in detail by classification results for hit monomials in $\mathbf{P}(2)$ and $\mathbf{P}(3)$, which can be found in Kameko's thesis [22] and more recently in Janfada's thesis [21].

Theorem 1.17 *The dimension of $Q^d(\mathbf{P}(n))$ is bounded independently of d for all n. The best bounds for $n = 1, 2, 3$ are respectively $1, 3, 21$.*

Kameko conjectures that the best bound for the case of $\mathbf{P}(n)$ is $(1)(3)\cdots(2^n - 1)$.

The following table, taken form [22, 2], shows the dimensions of $Q^d(\mathbf{P}(3))$ in terms of d.

Theorem 1.18 *The dimension of $Q^d(\mathbf{P}(3))$ is zero unless $d = 2^{s+t+u} + 2^{t+u} + 2^u - 3$, where $s \geq 0, t \geq 0, u \geq 0$. In this case the dimension is independent of u when $u > 0$ and depends on s, t as follows.*

dim	$u = 0$	$u \geq 0$				
$Q^d(\mathbf{P}(3))$	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s > 4$
$t = 0$	1	3	7	10	13	14
$t = 1$	3	8	15	14	14	14
$t = 2$	6	14	21	21	21	21
$t > 2$	7	14	21	21	21	21

The following result [8] provides a set of generators for $\mathbf{P}(n)$ as an $\mathcal{A}$-module but is not minimal.

Theorem 1.19 *The $\mathcal{A}$ module $\mathbf{P}(n)$ is generated by monomials $x_1^{e_1} \cdots x_n^{e_n}$ where, up to permutation of the variables, $\alpha(e_i + 1) \leq i$.*

This result leads to a proof of Theorem 1.17 [8] . We quote one final result [36] in this section which narrows down the scope of a minimal generating set and refines Theorem 1.16.

Theorem 1.20 *If a monomial in $\mathbf{P}^d(n)$ has ω-order less than that of a minimal spike in degree d, then f is hit. A generating set for $\mathbf{P}(n)$ can be chosen from monomials whose ω-order is between that of a minimal and maximal spike in any degree.*

There are degrees for which there is only one spike up to permutation of the variables. In such degrees d it can be verified that the dimension of $Q^d(\mathbf{P}(n))$ is bounded by the product $1(3)\cdots(2^n - 1)$ and a generating set can be written down. The difficulty in proving the Kameko conjecture in general seems to be in degrees where there are spikes of various ω-orders.

2 The symmetric hit problem

An element π in the symmetric group Σ_n acts on the right of a polynomial $f \in \mathbf{P}(n)$ by permuting the variables and the action is clearly multiplicative i.e. $(fg)\pi = (f\pi)(g\pi)$ for two polynomials $f, g \in \mathbf{P}(n)$. A hit equation in $\mathbf{B}(n)$ may always be taken, when convenient, in the form of a finite sum

$$a = \sum_{k \geq 0} Sq^{2^k} b_i,$$

where a and the b_i are symmetric polynomials.

As we shall explain in the next section, $\mathbf{B}(n)$ is generated additively by the symmetrised monomials. The ω-order and α-order are symmetric in the variables and apply therefore to monomial symmetric functions in $\mathbf{B}(n)$.

2.1 Symmetrisation

Given a monomial f in a subset of the variables $x_1, \ldots, x_n$, we can form the *symmetrisation* of f which means the smallest symmetric function $\sigma(f)$ in $\mathbf{B}(n)$ containing f as a term. To be more precise, let $\pi_1, \cdots, \pi_t$ be a set of left coset representatives for the stabiliser of the monomial f in Σ_n. Then $\sigma(f) = \sum_{i=1}^{t} f\pi_i$. For example, if the exponents of the variables $x_1, \ldots, x_n$ in f are all distinct then the stabiliser of f is trivial and $\sigma(f) = \sum_\pi f\pi$, where the summation is taken over the whole of Σ_n. This is the classical transfer of invariant theory. At the other extreme, if all exponents of f are the same then the stabiliser is the whole of Σ_n and $\sigma(f) = f$. If $\pi_1, \cdots, \pi_t$ are left coset representatives for a subroup of the stabiliser of f it is still true, of course, that $\sum_{j=1}^{t} f\pi_j$ is symmetric but the expression may be zero. It should be emphasised that the meaning of $\sigma(f)$ depends on the set of variables over which symmetrisation is taking place. For example $\sigma(x_1)$ means $x_1 + x_2$ in $\mathbf{P}(2)$ but $x_1 + x_2 + x_3$ in $\mathbf{P}(3)$. The symmetrised monomials form a vector space basis of $\mathbf{B}(n)$. In recent literature on invariant theory [37] the symmetrisation operator σ is referred to as the first Chern class.

What we would like to do is convert hit equations in $\mathbf{P}(n)$ into hit equations in $\mathbf{B}(n)$ by symmetrisation. The following example shows that a naive approach to this problem does not always work.

Example 2.1 *In $\mathbf{P}(2)$ we have the hit equation*

$$x_1^2 x_2^2 = Sq^1(x_1 x_2^2).$$

If we symmetrise this equation we obtain

$$0 = Sq^1(x_1 x_2^2 + x_1^2 x_2)$$

because we are working modulo 2. So we cannot prove this way that $x_1^2x_2^2$ is symmetrically hit. But, as it happens, there is a symmetric hit equation in $\mathbf{B}(2)$ *namely*

$$Sq^2(x_1x_2) = x_1^2x_2^2,$$

which shows that $x_1^2x_2^2$ is indeed symmetrically hit.

It is this phenomenon which prompts the questions raised in Problems 5.7 about whether the symmetrisation of a hit monomial in $\mathbf{P}(n)$ is symmetrically hit. There are examples of monomials which are not hit in $\mathbf{P}(n)$ but whose symmetrisations are hit in $\mathbf{B}(n)$. There are also examples of polynomials in $\mathbf{B}(n)$ which are hit in $\mathbf{P}(n)$ but not symmetrically hit in $\mathbf{B}(n)$.

However, there are circumstances in which we can symmetrise hit equations in $\mathbf{P}(n)$, based on the following observation.

Proposition 2.2 *Let f be a monomial and g a polynomial in $\mathbf{P}^d(n)$. Suppose there is a hit equation*

$$f - g = \sum_i \Theta_i f_i$$

in $\mathbf{P}^d(n)$, satisfying the condition that the stabiliser of f is a subgroup of the stabiliser of g and a subgroup of the stabiliser of each polynomial f_i. Let $\pi_1, \ldots, \pi_t$ be a collection of left coset representatives for the stabiliser of f in the symmetric group Σ_n. Then

$$\sigma(f) - \sum_{j=1}^{t} g\pi_j = \sum_i \Theta_i (\sum_{j=1}^{t} f_i\pi_j),$$

is a symmetric hit equation in $\mathbf{B}(n)$.

The reason is that $\sigma(f)$ is equal to $\sum_{j=1}^{t} f\pi_j$ by definition, and the expressions

$$\sum_{j=1}^{t} g\pi_j, \quad \sum_{j=1}^{t} f_i\pi_j$$

are symmetric by our earlier discussion about stabiliser subgroups. Under the given conditions we have the equivalence $\sigma(f) \cong \sum_{j=1}^{t} g\pi_j$ in $\mathbf{Q}(\mathbf{B}(n))$.

2.2 The symmetrised χ-trick

We now develop some useful symmetric hit equations by exploiting Proposition 2.2.

Proposition 2.3 *If the exponents of a hit monomial f in $\mathbf{P}(n)$ are all distinct. Then $\sigma(f)$ is symmetrically hit.*

In this case $\sigma(f)$ is the transfer of f and the stabiliser of f is the trivial group. The conditions of Proposition 2.2 are obviously satisfied with $g = 0$.

We can form a symmetrical version of Proposition 1.14

Proposition 2.4 *Let $f = uv^{2^k}$ be a monomial in $\mathbf{P}(n)$, with $\omega_j(u) = 0$ for $j \geq k$. Assume that u is symmetric and that $\sigma(v)$ is symmetrically hit. Then $\sigma(f)$ is symmetrically ω-reducible in $\mathbf{B}(n)$. In fact the ω-vector of the reduction can be assumed to be reduced in some position prior to k.*

To prove this result we note first of all that $\sigma(f) = u\sigma(v^{2^k})$ because u is symmetric. Since $\sigma(v)$ is symmetrically hit, the argument in the proof of Propostion 1.14 goes through when applied to $\sigma(v)$ in place of v, maintaining symmetry at each stage.

We continue with a symmetric analogue of Proposition 1.9.

Proposition 2.5 *Let $f = uv$ be a monomial in $\mathbf{P}(n)$ with the property that the stabiliser of f also stabilises u and v separately. Let $\pi_1, \ldots, \pi_t$ be a collection of left coset representatives for the stabiliser of f in the symmetric group Σ_n. Then for any element $\Theta \in \mathcal{A}$ there is a symmetric hit equation in $\mathbf{B}(n)$ of the form*

$$\sum_{j=1}^{t}(u\Theta(v))\pi_j - \sum_{j=1}^{t}(\widehat{\Theta}(u)v)\pi_j = \sum_{i>0} Sq^i \sum_{j=1}^{t}\sum_{l}(\Theta_{il}(u)\Phi_{il}(v))\pi_j,$$

for certain elements $\Theta_{il}, \Phi_{il} \in \mathcal{A}$. In particular we have the equivalence

$$\sum_{j=1}^{t}(u\Theta(v))\pi_j \cong \sum_{j=1}^{t}(\widehat{\Theta}(u)v)\pi_j$$

in $\mathbf{Q}(\mathbf{B}(n))$.

The proof follows immediateley from Propositions 1.9 and 2.2. We apply this result to obtain the symmetric analogue of Proposition 1.12.

Theorem 2.6 *Let $f = uv^{2^k} \in \mathbf{P}(n)$ be a monomial and suppose* $\deg(u) < (2^k - 1)\mu(\deg(v))$*. Then $\sigma(f)$ is symmetrically hit.*

To prove this statement, we first observe that the stabiliser of f also stabilises u and v. Hence Proposition 2.5 applies to the choice

$$\Theta = Sq^{2^{k-1}d} Sq^{2^{k-2}d} \cdots Sq^{2d} Sq^{d},$$

where d is the degree of v. The rest of the argument follows the same pattern as the proof of Theorem 1.12, noting that

$$\sigma(f) = \sum_{j=1}^{t}(u\Theta(v))\pi_j$$

for the particular choice of coset representatives π_j.

Before stating the next corollary, we need to make a few more remarks about symmetrisation. Consider a monomial f in $\mathbf{P}(n)$ expressed in the form $f = gh$ where g involves only the variables $x_1, \ldots, x_p$ and h involves only the remaining variables $x_{p+1}, \ldots, x_{p+q}$, where $p+q = n$. Let us suppose also that no exponent in the monomial g is equal to any exponent in the monomial h. Then the stabiliser of f is a subgroup of the cartesian product $\Sigma_p \times \Sigma_q$ which permutes the first p variables and last q variables separately. There is a set of left coset representatives for the stabiliser of f of the form $(\rho_i \times \tau_j)\zeta_k$, where the ρ_i run through a set of coset representatives for the stabiliser of g in Σ_p, the τ_j do the same for the stabiliser of h in Σ_q and the ζ_k are the shuffle permutations which preserve the orders of the two separate lists of variables $x_1, \ldots, x_p$ and $x_{p+1}, \ldots, x_{p+q}$. We then have the following lemma.

Lemma 2.7 *The symmetrisation of $f = gh$ in the n variables $x_1, \ldots, x_n$ is given by*

$$\sigma(f) = \sum_k (\sigma'(g)\sigma''(h))\zeta_k,$$

where σ', σ'' denote symmetrisation in the subsets $x_1, \ldots, x_p$ and $x_{p+1}, \ldots, x_{p+q}$ separately, and the ζ_k run through the shuffles of the first set in the second set.

The next corollary may be viewed in terms of a horizontal splitting of a block.

Proposition 2.8 *Let $f = gh$ in $\mathbf{P}(n)$ be a monomial factorised such that g involves only the variables $x_1, \ldots, x_p$ and h involves only the remaining variables $x_{p+1}, \ldots, x_{p+q}$, where $p + q = n$. Assume also that no exponent in the monomial g is equal to any exponent in the monomial h. Suppose there is a symmetric hit equation in $\mathbf{B}(q)$ of the form*

$$\sigma''(h) = \sum_r \Omega_r h_r,$$

for positively graded elements Ω_r in the Steenrod algebra. Then there is a symmetric equivalence in $\mathbf{B}(n)$ of the form

$$\sigma(f) \cong \sum_k \sum_r (\widehat{\Omega_r}(\sigma'(g))h_r)\zeta_k.$$

The proof of this result follows the line of argument in Proposition 2.5, once we observe that the stabiliser of f must stabilise g and h individually because these monomials have no exponents in common. We have

$$\sigma'(g)\sigma''(h) = \sum_r \sigma'(g)\Omega_r h_r = \sum_r (\widehat{\Omega_r}(\sigma'(g))h_r + \sum_r \sum_{i>0} Sq^i(\sum_k (\Theta_{ikr}(u)\Phi_{ikr}(v)),$$

for certain elements Θ_{ikr}, Φ_{ikr} in the Steenrod algebra. Now all terms in this equation are stabilised by the stabiliser of f. It follows that an application of the shuffle operators ζ_k to both sides of the equation, and adding over k, produces a symmetric hit equation. By Lemma 2.7 the left hand side becomes $\sigma(f)$ and the result is established.

2.3 Results for $\mathbf{B}(n)$

We now state answers to some of the basic Problems 1.1 about $\mathbf{B}(n)$ in parallel with the corresponding answers for $\mathbf{P}(n)$. First of all, the Peterson conjecture remains true for $\mathbf{B}(n)$, with the same condition as for $\mathbf{P}(n)$.

Theorem 2.9 *The quotient space* $\mathbf{Q}^d(\mathbf{B}(n))$ *is zero if and only if* $\mu(d) > n$.

The proof is an immediate consequence of Theorem 2.6 and the arguments used in the proof of Theorem 1.16.

It should be emphasised what this result actual says. It follows immediately from Theorem 1.16 that a homogeneous symmetric polynomial is hit in $\mathbf{P}(n)$ if its degree d satisfies $\mu(d) > n$. It is not so obvious, however, why this should imply that the polynomial is symmetrically hit in these degrees, in other words why it should be hit as an element of the module $\mathbf{B}(n)$. It was only by constructing the specially adapted hit equations that we were able to prove the symmetrised Peterson conjecture. This leads once again to the more general questions about symmetric hit elements posed in Problems 5.7.

For small values of n we have an analogue of Theorem 1.17.

Theorem 2.10 *For* $n = 1, 2, 3$, *the best upper bounds for the dimension of* $\mathbf{Q}^d(\mathbf{B}(n))$ *are respectively* $1, 1, 4$.

We ask in Problem 5.8 for a bound for the dimension of $\mathbf{Q}^d(\mathbf{B}(n))$ in general and some sensible upper estimate of it, analogous to the Kameko conjecture.

As far as minimal bases are concerned, the situation for $n = 1$ is straightforward since $\mathbf{B}(1) = \mathbf{P}(1)$ and $\mathbf{B}(1)$ is generated by the spikes $x_1^{2^\epsilon - 1}$.

In the 2-variable case, the answer is also quite simple.

Theorem 2.11 *The collection of symmetric functions*

$$x_1^{2^r-1}x_2^{2^r-1}, \quad x_1^{2^s-1}x_2^{2^t-1} + x_1^{2^t-1}x_2^{2^s-1},$$

for $r \geq 0$ *and* $s > t \geq 0$, *forms a minimal generating set for* $\mathbf{B}(2)$.

Theorem 2.11 indicates that the symmetrised spikes are enough to furnish a generating set of the module in the case of two variables. The situation for three variables is more complicated. We provide a table, by analogy with Theorem 1.18, showing the dimensions of $\mathbf{Q}^d(\mathbf{B}(3))$ in degrees d where the dimensions are non-zero.

Theorem 2.12 *The dimension of $\mathbf{Q}^d(\mathbf{B}(3))$ is zero unless $d = 2^{s+t+u} + 2^{t+u} + 2^u - 3$, where $s \geq 0, t \geq 0, u \geq 0$. In this case the dimension is independent of u when $s > 0$ and depends on s, t as follows.*

dim	$u = 0$	$u \geq 0$			
$\mathbf{Q}^d(\mathbf{B}(3))$	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s > 3$
$t = 0$	1	1	2	3	4
$t > 0$	1	1	2	2	2

The form of the degree d in the above theorem implies of course that $\mu(d) \leq 3$ in accordance with Theorem 2.9. The next statement exhibits a list of minimal generators for $\mathbf{B}(3)$.

Theorem 2.13 *A minimal generating set for $\mathbf{B}(3)$ as a module over the Steenrod algebra consists of the symmetrised spikes together with the symmetrisations of monomials $x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3}$ of three types.*

$$\epsilon_1 = 2^{u+s-1} - 1, \quad \epsilon_2 = \epsilon_3 = 2^u + 2^{u+s-2} - 1, \qquad \text{for } u \geq 0, s > 2$$

$$\epsilon_1 = 2^{u+2} - 1, \quad \epsilon_2 = \epsilon_3 = 2^{u+s-1} - 2^u - 1, \qquad \text{for } u \geq 0, s > 3$$

$$\epsilon_1 = 2^u - 1 + 2^{u+t+s-1}, \quad \epsilon_2 = 2^{u+t} - 1, \quad \epsilon_3 = 2^{u+t+s-1} - 1 \qquad \text{for } u \geq 0, t > 0, s > 1$$

The three types of monomials exhibited in Theorem 2.13 can be visualised in terms of their binary block diagrams as exhibited below.

$$A_1 = \begin{matrix} 1 & - & 1 & 1 & - & 1 & 1 \\ 1 & - & 1 & 0 & - & 0 & 1 \\ 1 & - & 1 & 0 & - & 0 & 1 \end{matrix} \qquad A_2 = \begin{matrix} 1 & - & 1 & 1 & 1 & & & \\ 1 & - & 1 & 0 & 1 & 1 & - & 1 \\ 1 & - & 1 & 0 & 1 & 1 & - & 1 \end{matrix}$$

$$A_3 = \begin{matrix} 1 & - & 1 & 0 & - & 0 & 0 & - & 0 & 1 \\ 1 & - & 1 & 1 & - & 1 & & & & \\ 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & \end{matrix}$$

The first two monomials A_1, A_2 lie in degree $2^{u+s} + 2^{u+1} - 3$ for $u \geq 0, s > 2$, where there are also two symmetrised spikes. The third monomial A_3 lies in degree $2^{u+t+s} + 2^{u+t} + 2^u - 3$ for $u \geq 0, t > 0, s > 1$, where there is just one symmetrised spike. There are alternative sets of generators for $\mathbf{B}(3)$ which serve various purpose. For more details on the classification of symmetrised monomials into hits and non-hits, in answer to question 1 of Problems 1.1 we refer to Janfada's thesis [21].

2.4 The submodules $\mathbf{M}(n)$

We note that $\mathbf{B}(n)$ splits as a module over $\mathcal{A}$ into a direct sum of certain submodules $\mathbf{T}(r)$ for $r \leq n$. To be precise

$$\mathbf{B}(n) = \bigoplus_{r=1}^{n} \mathbf{T}(r),$$

where $\mathbf{T}(r)$ is generated by the symmetrisation of monomials involving precisely r of the n variables $x_1, \ldots x_n$. One sees therefore that $\mathbf{T}(r)$ is isomorphic to $\mathbf{M}(r)$ as a module over the Steenrod algebra and we deduce the following dimension formulae.

Proposition 2.14 $\dim \mathbf{Q}^d(\mathbf{M}(n)) = \dim \mathbf{Q}^d(\mathbf{B}(n)) - \dim \mathbf{Q}^d(\mathbf{B}(n-1))$

$$\dim \mathbf{Q}^d(\mathbf{B}(n)) = \sum_{r=1}^{n} \dim \mathbf{Q}^d(\mathbf{M}(r)).$$

There are topological problems [39, 29] associated with these algebraic statements that we shall discuss later.

3 Modular representations and hit problems

As general references for this section we cite [3, 15, 16, 17, 27, 28, 44].

There are 2^n distinct irreducible modular representations of $M(n, \mathbb{F}_2)$ over the natural field $\mathbb{F}_2$, parametrised by sequences of non-negative integers

$$\lambda = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0,$$

subject to the constraints $\lambda_i - \lambda_{i+1} \leq 1$ for $1 \leq i < n$ and $\lambda_n \leq 1$, called 2-*column regularity*. The irreducible representations of $GL(n, \mathbb{F}_2)$ correspond to those λ with $\lambda_n = 0$.

In the literature on the representation theory of symmetric groups and general linear groups, the sequence λ is usually referred to as a partition of length n of the number $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and depicted by a *Ferrers diagram*, which is a matrix with a mark at each position (i, j) for $1 \leq j \leq \lambda_i$, and other positions empty. The non-empty positions are called the 'nodes' of the diagram. The transpose of the Ferrers diagram of λ corresponds to the *conjugate* sequence λ', where λ'_i is the number of rows k such that $\lambda_k \geq i$. The sequence λ' is again a partition of $|\lambda| = |\lambda'|$ but does not necessarily correspond to an irreducible representation of $M(n, \mathbb{F}_2)$. If λ is 2-column regular, then λ' is strictly monotonic. Such sequences are also referred to in the literature as 2-regular. For example, the diagram of $\lambda = (3, 3, 2, 2, 1)$ and its transpose are shown below.

$$\lambda = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & \\ 1 & 1 & \\ 1 & & \end{matrix} \qquad \lambda' = \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & & & \end{matrix}$$

The nodes of a Ferrers diagram are sometimes replaced by squares in which other information is inserted.

The empty diagram corresponds to the trivial representation of $M(n, \mathbb{F}_2)$, where every matrix acts as the identity. The Ferrers diagram with just one entry is the natural representation of $n \times n$ matrices on n-dimensional vectors. More generally, the diagram with just one column containing r entries is the r-th exterior power of the natural representation. In particular, when $r = n$, we obtain the determinant representation in which all non-singular matrices act as the identity but singular matrices act as 0. The Ferrers diagram with the maximal allowable number of entries, corresponding to the triangular partition $(n, n-1, \ldots, 1)$, is referred to as the Steinberg representation for the semigroup $M(n, \mathbb{F}_2)$. The partition $(n-1, \ldots, 1, 0)$ is the Steinberg representation for the group $GL(n, \mathbb{F}_2)$. If a Ferrers diagram with $\lambda_n = 0$ is interpreted as a representation of $GL(n, \mathbb{F}_2)$, then adding 1 to each λ_i produces a full Ferrers diagram (now $\lambda_n = 1$), which corresponds to tensoring with the determinant representation, where the singular matrices now act as 0. Conversely, a representation of $M(n, \mathbb{F}_2)$ corresponding to a full Ferrers diagram may be interpreted in this way, as arising from a representation of $GL(n, \mathbb{F}_2)$. Of course any Ferrers diagram with $\lambda_n = 0$ may also be interpreted as a representation of $M(n, \mathbb{F}_2)$ but it is not obvious how this is related to the group interpretation, except in cases like the natural represention. For more explanation on this point see Harris and Kuhn [17].

For further discussion of the representation theory of $M(n, \mathbb{F}_2)$ we need to work with the equivalent theory of the representations of the semigroup algebra

$$\mathbb{F}_2[M(n, \mathbb{F}_2)].$$

An element e in an algebra is called *idempotent* if $e^2 = e$. The elements $0, 1$ are the *trivial* idempotents. Two idempotents e, f are *orthogonal* if $ef = fe = 0$. An idempotent e is primitive if it cannot be written as a sum two non-trivial orthogonal idempotents. The idempotent is *central* if it commutes with all elements of the algebra. A central idempotent is *centrally primitive* if it is not the sum of non-trivial orthogonal central idempotents. In the finite dimensional algebras that concern us, there is a uniquely determined finite set of centrally primitive idempotents whose sum is the identity of the algebra.

Example 3.1 *There are three centrally primitive idempotents in the semigroup algebra* $\mathbb{F}_2[M(n, \mathbb{F}_2)]$.

$$z_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} z_1 = {} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ & + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} z_2 = {} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ & + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

Then we see that

$$1 = z_0 + z_1 + z_2$$

and it can be checked that these idempotents are orthogonal. The display is arranged to highlight the sums of the non-singular matrices in z_1 *and* z_2*, which provide the two centrally primitive idempotents in the group algebra* $\mathbb{F}_2[GL(2, \mathbb{F}_2)]$.

In the semigroup algebra $\mathbb{F}_2[M(n, \mathbb{F}_2)]$ each central idempotent decomposes into a sum of orthogonal primitive idempotents. This decomposition is not unique. It turns out that the conjugacy classes of the primitive idempotents are in bijective correspondence with the irreducible representations λ. Let $\delta(\lambda)$ denote the dimension of the representation associated with the partition λ. Then there is a decomposition of the identity of the algebra into a maximal set of orthogonal primitive idempotents

$$1 = \sum_{\lambda} \sum_{i=1}^{\delta(\lambda)} e_{\lambda,i},$$

where the $e_{\lambda,i}$, for $1 \leq i \leq \delta(\lambda)$, is a set of conjugate primitive orthogonal idempotents associated with the same irreducible representation λ. It is customary to write, somewhat inaccurately,

$$1 = \sum_{\lambda} \delta(\lambda) e_{\lambda},$$

where e_λ stands for a typical idempotent associated with λ. Although the decomposition of 1 into orthogonal primitive idempotents is not unique, any two such sets of idempotents are conjugate by an invertible element in the semigroup ring in a way that matches idempotents associated with the same irreducible representation. In particular, in any decomposition of 1 into orthogonal primitive idempotents, there will be some way of grouping the idempotents into subsets which add up to the central idempotents but exactly how this works is a complicated matter to do with block theory of modular representations. We content ourselves here with an illustration of how it works for one particular decomposition in the case of 2×2 matrices over $\mathbb{F}_2$.

Example 3.2 *There are six elements in any choice of a maximal set of primitive orthogonal idempotents decomposing* 1 *in* $\mathbb{F}_2[M(n, \mathbb{F}_2)]$. *Such a set can be chosen to refine the central idempotents in the following way.*

$$e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$e_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$e_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$e'_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have the following decompositions of the central idempotents

$$z_0 = e_0, \quad z_1 = e_1 + e'_1 + e_{11}, \quad z_2 = e_{21} + e'_{21}$$

and the maximal decomposition of the identity

$$1 = e_0 + e_1 + e'_1 + e_{11} + e_{21} + e'_{21}.$$

Note that the e_0, e_1, e'_1 are the singular parts of e_{11}, e_{21}, e'_{21}. On the other hand, the non-singular parts g_0, g_1, g'_1 of e_{11}, e_{21}, e'_{21} provide a maximal set of three orthogonal primitive idempotents for the group algebra $\mathbb{F}_2[GL(2, \mathbb{F}_2)]$.

The idempotent e_0 corresponds to the trivial representation of $\mathbb{F}_2[M(n, \mathbb{F}_2)]$ with empty Ferrers diagram. The conjugate idempotents e_1 and e'_1 are associated with the natural representation, e_{11} with the determinant representation and the conjugate idempotents e_{21}, e'_{21} with the Steinberg representation. For $GL(2, \mathbb{F}_2)$, the idempotent g_0 corresponds to the trivial representation and g_1, g'_1 are conjugate idempotents corresponding to the Steinberg representation, which happens to coincide with the natural representation in case $n = 2$.

The action of $M(n, \mathbb{F}_2)$ on $\mathbf{P}(n)$ extends naturally to an action of the semigroup ring $\mathbb{F}_2[M(n, \mathbb{F}_2)]$ and the idempotents induce a corresponding decomposition of the polynomial algebra

$$\mathbf{P}(n) = \oplus_\lambda \delta(\lambda) \mathbf{P}(n) e_\lambda,$$

compatible with the left action of the Steenrod algebra. Each 'piece' $\mathbf{P}(n)e_\lambda$ occurs $\delta(\lambda)$ times in the decomposition and is an indecomposable $\mathcal{A}$-submodule (but no longer a right $\mathbb{F}_2[M(n, \mathbb{F}_2)]$-module). The dimension

$$\nu_d(\lambda) = \dim(\mathbf{P}^d(n) e_\lambda)$$

is the number of occurrencies of λ as a composition factor in $\mathbf{P}^d(n)$. One can think of the action of e_λ on $\mathbf{P}^d(n)$ as picking off a 1-dimensional vector subspace of $\mathbf{P}^d(n)$ for each occurrence of λ as a composition factor.

One way of tackling the hit problem for $\mathbf{P}(n)$ over $\mathcal{A}$ is to solve the hit problem for each piece separately. If an element of $\mathbf{P}^d(n)e_\lambda$ is hit in $\mathbf{P}(n)$ then it is already hit in $\mathbf{P}^d(n)e_\lambda$, as can be seen by applying the idempotent to both sides of a hit equation in $\mathbf{P}(n)$. This approach to the hit problem demands good control over the idempotents, which is lacking in general, but there are some interesting particular cases where progress can be made. In particular, we see that the central idempotents preserve symmetric functions and induce a decomposition of $\mathbf{B}(n)$ into $\mathcal{A}$-module summands. By following through the action of the idempotent splitting in Example 3.2 on symmetric functions, we obtain a splitting of $\mathbf{B}(2)$ into four $\mathcal{A}$-submodules

$$\mathbf{B}(2) = \mathbf{B}(2)e_0 \oplus \mathbf{B}(2)e_1 \oplus \mathbf{B}(2)e_{11} \oplus \mathbf{B}(2)e_{21},$$

where $\mathbf{B}(2)e_1$ is isomorphic as an $\mathcal{A}$-module to $\mathbf{B}(1)$, generated by $x_1^a + x_2^a$ for $a > 0$: the module $\mathbf{B}(2)e_{11}$ is isomorphic to the Dickson algebra $\mathbf{D}(2)$, generated by

$$(x_1 + x_2)^a(x_1^b + x_2^b) + (x_1 + x_2)^b(x_1^a + x_2^a) + x_1^a x_2^b + x_1^b x_2^a,$$

for $a, b > 0$: the module $\mathbf{B}(2)e_{21}$ is generated by

$$(x_1 + x_2)^a(x_1^b + x_2^b) + (x_1 + x_2)^b(x_1^a + x_2^a),$$

for $a, b > 0$. We note that $\mathbf{B}(2)e_1', \mathbf{B}(2)e_{21}'$ are 0 and $\mathbf{B}(2)e_0$ is trivial, concentrated in dimension 0. By restricting to symmetric functions divisible by by x_1x_2 we obtain a splitting of the A-module $\mathbf{M}(2)$ into two pieces

$$\mathbf{M}(2) = \mathbf{M}(2)e_{11} \oplus \mathbf{M}(2)e_{21},$$

where $\mathbf{M}(2)e_{11}$ is the Dickson algebra $\mathbf{D}(2)$.

It is debatable whether the attempt to solve the hit problem for $\mathbf{B}(2)$ by decomposition methods is any better than the direct approach in arriving at Theorem 2.11 but it does raise a number of interesting Problems 5.9, 5.12 and 5.11 about the algebraic splittings of $\mathbf{P}(n), \mathbf{B}(n), \mathbf{M}(n)$ and hit problems for individual pieces.

We now turn to some topological aspects of the problem.

3.1 Modular representations and topological splittings

We use the notation $L(\lambda)$ for a $M(n, \mathbb{F}_2)$-module which affords the irreducible representation corresponding to λ. It is known that $L(\lambda)$ occurs as a composition factor in $\mathbf{P}^d(n)$ for some value of d. Indeed, it actually occurs as a submodule of $\mathbf{P}^d(n)$ for some (usually higher) value of d. The following statements indicate when these phenomena first happen [6].

Theorem 3.3 *The irreducible $M(n, \mathbb{F}_2)$-representation corresponding to the 2-column regular partition λ occurs for the first time as a composition factor in $\mathbf{P}^d(n)$ in degree*

$$\zeta(\lambda) = \sum_i 2^{\lambda_i} - 1$$

and for the first time as a submodule in $\mathbf{P}^d(n)$ in degree

$$\eta(\lambda) = \sum_i \lambda_i 2^{i-1}.$$

In the above formulae we note that the $\zeta(\lambda) < \eta(\lambda)$ except when λ is a triangular sequence $(m, m-1, \ldots, 2, 1)$. Note also that $\eta(\lambda) = \zeta(\lambda')$.

Since a singular matrix must annihilate some non-zero element in $\mathbf{P}^d(n)$ for $d > 0$, it follows that the trivial representation of $M(n, \mathbb{F}_2)$ can only occur once, namely in dimension 0. This is consistent with the fact that the trivial representation corresponds to the empty Ferrers diagram where all $\lambda_i = 0$, in which case $\zeta = \eta = 0$. On the other hand the determinant representation, where $\lambda_i = 1$ for $1 \leq i \leq n$, occurs for the first time in degree $\zeta = n$ as a composition factor headed by the product of the variables $x_1 \cdots x_n$. As a submodule it appears for the first time in degree $\eta = 2^n - 1$.

Little is known about the odd prime analogue of the first occurrence problem as a composition factor, although a few cases are resolved [4]. The first occurrence as a submodule is known for all primes [25]. Even where we have explicit models for the irreducible representations of $M(n, \mathbb{F}_2)$ as submodules of $\mathbf{P}(n)$, there seems to be no known closed formulae for their dimensions.

We now explain how the numbers $\delta(\lambda), \nu_d(\lambda), \zeta(\lambda), \eta(\lambda)$ can be interpreted topologically. For an early reference on the use of idempotents in splitting suspended spaces we cite [9].

Recall that $\mathbf{P}(n)$ is the cohomology of the product of n copies of infinite real projective space, otherwise known as the classifying space $B(\mathbf{Z}/2)^n$ of the group $(\mathbf{Z}/2)^n$. Let Y denote the suspension of $B(\mathbf{Z}/2)^n$. For each irreducible representation λ of $M(n, \mathbb{F}_2)$ there is an associated topological space Y_λ such that, up to homotopy type, Y decomposes into the one-point union

$$Y \simeq \vee_\lambda \delta(\lambda) Y_\lambda,$$

each Y_λ occurring $\delta(\lambda)$ times in the splitting. The cohomology $\mathbf{H}^*(Y_\lambda, \mathbb{F}_2)$ can be identified with $\mathbf{P}(n)e_\lambda$, with a shift in grading. In particular the dimension of $\mathbf{H}^d(Y_\lambda, \mathbb{F}_2)$ is $\nu_d(\lambda)$ and $\zeta(\lambda)$ corresponds to the connectivity of the piece Y_λ. None of the pieces Y_λ can be further split stably into a one-point union of non-trivial spaces. The piece associated with the idempotent corresponding to the trivial representation, given by the empty Ferrers diagram, is a single point. In practice, therefore, there are $2^n - 1$ interesting spaces in the splitting of Y.

We refer to [15, 27, 28, 33] for a detailed analysis of the topological pieces obtained for the case $n = 2$ in the stable splittings of $B(\mathbb{Z}/2)$ and $BO(2)$. In the terminology of Example 3.2 there is a stable homotopy equivalence

$$Y \simeq Y_0 \vee 2Y_1 \vee Y_{11} \vee 2Y_{21}.$$

We have topological Problems 5.15, 5.16, analogous to the algebraic Problems 5.9, 5.12.

We now look at a few particular problems related to the general discussion above.

3.2 Linking first occurences by Steenrod operations

In this section we shall describe an explicit Steenrod operation which links the first occurrence of the irreducible representation λ as composition factor with its first occurrence as submodule in $\mathbf{P}(n)$. We need some preliminary notation.

In the Ferrers diagram of λ, the kth *anti-diagonal* of λ consists of the nodes (i, j) such that $i + j = k + 1$. Suppose the nodes of the k-th anti-diagonal are $(k, 1), (k-1, 2), \ldots, (k-s+1, s)$. The associated van der Monde determinant is defined by

$$v_k(\lambda) = \begin{vmatrix} x_k & x_{k-1} & \cdots & x_{k-s+1} \\ x_k^2 & x_{k-1}^2 & \cdots & x_{k-s+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{2^{s-1}} & x_{k-1}^{2^{s-1}} & \cdots & x_{k-s+1}^{2^{s-1}} \end{vmatrix}.$$

The product of these expressions is denoted by

$$v(\lambda) = \prod_k v_k(\lambda).$$

For example, when $\lambda = (2, 1, 1)$,

$$v(\lambda) = x_1 \cdot [x_1, x_2^2] \cdot x_3 = x_1^3 x_2 x_3 + x_1^2 x_2^2 x_3.$$

In general, the 'leading' term of $v(\lambda)$, i.e. the monomial with highest exponents in the left lexicograhic order, is $\prod_k x_k^{2^{\lambda_k}-1}$, which is a spike. The polynomial $v(\lambda)$ is therefore not hit.

We shall also use following notation for the particular van der Monde determinant

$$w(n) = [x_1, x_2^2, \cdots, x_n^{2^{n-1}}] = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2^{n-1}} & x_2^{2^{n-1}} & \cdots & x_n^{2^{n-1}} \end{vmatrix},$$

where the shorthand form in square brackets lists just the diagonal elements of the determinant. Then we associate with λ the polynomial

$$w(\lambda) = \prod_k w_k(\lambda_k).$$

For example, when $\lambda = (2, 1, 1)$,

$$w(\lambda) = [x_1, x_2^2]x_2x_3, \qquad w(\lambda') = [x_1, x_2^2, x_3^4]x_1.$$

Note that $\deg(v(\lambda)) = \eta(\lambda), \deg(w(\lambda)) = \zeta(\lambda)$. We now state the main results [41].

Theorem 3.4 *Let λ be a 2-column regular partition of length n. Then the corresponding irreducible $M(n, \mathbb{F}_2)$-module $L(\lambda)$ appears as a top composition factor of the module generated in $\mathbf{P}_{\zeta(\lambda)}(n)$ by $v(\lambda)$ and also as the submodule in $\mathbf{P}_{\eta(\lambda)}(n)$ generated by $w(\lambda')$.*

Theorem 3.5 *Let λ be a 2-column regular partition of length n. For $1 \leq k \leq \lambda_1$, let $r_k = (2^{\lambda'_k} - 1) - \sum_{i \leq \lambda'_k} 2^{\lambda_i - k}$. Then*

$$\chi(Sq^{r_1}Sq^{r_2} \ldots Sq^{r_{\lambda_1}})v(\lambda) = w(\lambda').$$

The sequence of numbers $(r_1, r_2, \ldots, r_{\lambda_1})$ can be conveniently calculated from the tableau obtained by inserting integers into the Ferrers diagram of λ as follows: if the (i, λ_i) is the highest node in its antidiagonal, insert $2^{i-1} - 1$ in that position and continue down the diagonal by doubling the number entered at each step. The sum of the numbers entered in column k is then r_k.

Example 3.6 For $\lambda = (3, 3, 2, 2, 1)$ we obtain $(r_1, r_2, r_3) = (18, 9, 1)$ using the tableau shown below.

0	0	0
0	0	1
0	2	
4	7	
14		

The statement of Theorem 3.5 in this case is

$$\chi(Sq^{18}Sq^9Sq^1)(x_1 \cdot [x_1, x_2^2] \cdot [x_1, x_2^2, x_3^4] \cdot [x_2, x_3^2, x_4^4] \cdot [x_4, x_5^2])$$
$$= [x_1, x_2^2, x_3^4, x_4^8, x_5^{16}] \cdot [x_1, x_2^2, x_3^4, x_4^8] \cdot [x_1, x_2^2].$$

3.3 The Steinberg piece

The Steinberg representation of $GL(n, \mathbb{F}_2)$, corresponding to the triangular sequence

$$St = (n-1, \ldots, 1, 0)$$

plays a special role in the representation theory of the general linear group. Every occurrence of St is a submodule. The following statement solves the hit problem for the Steinberg piece [41] arising from the general linear group.

Theorem 3.7 *There is a choice of indecomposable idempotent e_{St} in the group algebra $\mathbb{F}_2[GL(n, \mathbb{F}_2)]$ associated with the Steinberg representation St of $GL(n, \mathbb{F}_2)$ such that the piece $\mathbf{P}(n)e_{St}$ is generated minimally by the symmetrised spikes*

$$\sigma(x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}),$$

for distinct *exponents $d_1, d_2, \ldots, d_n$.*

The actual choice of idempotent e_{St} is constructed as follows. Let $\overline{U}_n$ denote the sum of the upper triangular matrices in $GL(n, \mathbb{F}_2)$ and $\overline{\Gamma}_n$ the sum of the elements in the symmetric group Σ_n. Then

$$e_{St} = \overline{U}_n \overline{\Sigma}_n.$$

By construction we see that the action of this particular choice of Steinberg idempotent on $\mathbf{P}(n)$ preserves symmetric functions and therefore splits $\mathbf{B}(n)$ into a Steinberg piece and another piece. We refer to [27, 28, 33] for the topological realisation of this splitting and to [39, 29] for the stable splitting of $BO(n)$ into pieces corresponding to the submodules $\mathbf{T}(r)$ mentioned in Proposition 2.4.

3.4 The trivial piece

Recently, Hung and Nam [19] have proved Hung's conjecture that all elements in the Dickson algebra $\mathbf{D}(n)$ are hit in $\mathbf{P}(n)$ for $n \geq 3$. Now the Dickson algebra is only a part of the piece $\mathbf{P}(n)g_0$, corresponding to the trivial representation of the group $GL(n, \mathbb{F}_2)$. The Dickson algebra affords the *submodule* occurrences of the trivial representation in $\mathbf{P}(n)$. The hit problem for the Dickson algebra itself is difficult and has only been solved for small values of n [20]. It would be interesting to give a a minimal generating set for $\mathbf{P}(n)g_0$ by analogy with the Steinberg case. The Hung-Nam result says that all submodule occurences are hit by earlier composition factors in the determinant piece at least for $n \geq 3$.

We saw earlier how, in the case $n = 2$, the idempotent e_{11} splits off the Dickson algebra $\mathbf{D}(2)$ from $\mathbf{M}(2)$. Now $\mathbf{D}(2)$ is topologically realisable by $H^*(BSO(3))$ over the field of two elements. This raises again the question concerning the topological splitting of Thom complexes $MO(n)$.

4 The hit problem for the differential operator algebra

The action of the Steenrod square Sq^k on $\mathbf{P}(n)$ can be lifted to the action of an operator SQ^k on the polynomial algebra

$$\mathbf{W}(n) = \mathbb{Z}[x_1, x_2, \ldots, x_n]$$

over the integers. Integral squaring operators are members of a larger ring of operators $\mathcal{D}$, called the *differential operator algebra.* The formal definition of $\mathcal{D}$ and some of its properties can be found in [49, 48]. Topologists know $\mathcal{D}$ as the Landweber-Novikov algebra. For present purposes we recall from [49, 48] some of the main features concerning the action of $\mathcal{D}$ on $\mathbf{W}(n)$. An additive basis for $\mathcal{D}$ is formed from *wedge* products of the primitive partial differential operators

$$D_k = \sum_{i \geq 1} x_i^{k+1} \frac{\partial}{\partial x_i},$$

for $k \geq 1$, acting in the usual on $\mathbf{W}(n)$. Although D_k is formally an infinite sum, its action on a polynomial involves only a finite number of variables in any instance. The wedge product $\vee$ of two differential operators, with variable coefficients, is defined by allowing the derivatives of the first operator to pass the variable coefficients of the second operator without acting. The wedge product is commutative and gives the term of highest differential order in the composition of the operators. For example, the composite $D_1 \circ D_1$ is given by

$$(\sum_{i \geq 1} x_i^2 \frac{\partial}{\partial x_i})(\sum_{i \geq 1} x_i^2 \frac{\partial}{\partial x_i}) = 2(\sum_{i \geq 1} x_i^3 \frac{\partial}{\partial x_i}) + \sum_{(i_1, i_2)} x_{i_1}^2 x_{i_2}^2 \frac{\partial^2}{\partial x_{i_1} \partial x_{i_2}},$$

where the last summation is taken over all 2-vectors of non-negative integers (i_1, i_2). Hence $D_1 \circ D_1 = 2D_2 + D_1 \vee D_1$. It should be noted that $D_1 \vee D_1$ is divisible by 2 as an integral operator. More generally, an iterated wedge product is given by the formula

$$D_{k_1} \vee D_{k_2} \vee \cdots \vee D_{k_r} = \sum_{(i_1, \ldots, i_r)} x_{i_1}^{k_1+1} \cdots x_{i_r}^{k_r+1} \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}},$$

where the summation is taken over all r-vectors of non-negative integers. It can be seen from this that the iterated wedge product $D_k^{\vee r}$ is divisible by $r!$ as an integral operator. By definition, $\mathcal{D}$ is generated over the integers by the divided operators $D_k^{\vee r}/r!$ under wedge product. For convenience we use the multiset notation $K = k_1^{r_1} k_2^{r_2} \ldots k_a^{r_a}$ to denote a set of positive distinct integers k_i repeated r_i times. Then

$$D(K) = \frac{D_{k_1}^{\vee r_1}}{r_1!} \vee \frac{D_{k_2}^{\vee r_2}}{r_2!} \vee \cdots \vee \frac{D_{k_a}^{\vee r_a}}{r_a!}$$

denotes the iterated wedge product of divided differential operators. For example,

$$D(k) = D_k, \quad D(k^r) = \frac{D_k^{\vee r}}{r!}.$$

The collection $D(K)$, as K ranges over multisets of distinct integers, forms an additive basis for $\mathcal{D}$. A significant fact is that $\mathcal{D}$ is closed under composition of operators. Furthermore, the natural coproduct $\psi(D_n) = 1 \otimes D_n + D_n \otimes 1$ makes $\mathcal{D}$ into a Hopf algebra with respect to both the composition and the wedge products.

We define the *integral Steenrod squares* by $SQ^r = D(1^r)$. It is shown in [49] that the modulo 2 reduction of SQ^k is Sq^k. For example

$$SQ^1 = D(1) = \sum_{i \geq 1} x_i^2 \frac{\partial}{\partial x_i}.$$

Additively, the Steenrod algebra is generated by the modulo 2 reductions of those $D(K)$ for which the elements K have the form $k_i = 2^{\lambda_i} - 1$. This is called the *Milnor basis* of the Steenrod algebra.

Since $\mathcal{D}$ is defined over the integers we have the possibility of reduction at any prime as well as rational reduction. For example, the collection of modulo p reductions of those $D(K)$ for which the elements of K have the form $k_i = p^{\lambda_i} - 1$ constitute the Milnor basis of the Steenrod algebra of p-th power operators at an odd prime p. The analogue of $SQ^r = D(1^r)$ in the odd prime case is $P^r = D((p-1)^r)$.

We can pose the hit Problem 5.19 for the action of the differential operator algebra $\mathcal{D}$ on $\mathbf{W}(n)$ over the integers, but this would seem to be a very difficult question to answer in more than a few variables. For $n = 1$ the answer is simple because $D_k(x) = x^{k+1}$. Hence $\mathbf{Q}(\mathbf{W}(1))$ has rank 1 generated by x_1. This result generalises in the following way. Note first of all that the action of $\mathcal{D}$ commutes with the right action of the symmetric group because the differential operators are themselves symmetric in the variables and partial derivatives. On the other hand it does not commute with the action of all matrices over the integers. We lose the analogue of the Dickson algebra but retain the representation theory of the symmetric group. In particular we can study the hit problem for symmetric polynomials over the integers, viewed as a $\mathcal{D}$-module.

Theorem 4.1 *Any symmetric polynomial in $\mathbf{W}(n)^{\Sigma_n}$ divisible by $x_1 \cdots x_n$ and of degree strictly greater than n is hit by a differential operator in $\mathcal{D}$.*

Problems 5.20 remain for representations of the symmetric group other than the trivial one. Since integral representation theory of Σ_n is difficult, we look instead at modular and rational reductions.

4.1 The hit problem for $\mathcal{D}$ modulo 2

Integral results about the action of $\mathcal{D}$ on $\mathbf{W}(n)$ can be passed down to modular reductions. For example the statement of Theorem 4.1 is true for the action of $\mathcal{D} \otimes \mathbb{F}_2$ on $\mathbf{P}(n)$. As observed earlier $\mathcal{D} \otimes \mathbb{F}_2$ contains $\mathcal{A}$ as a sub-algebra. To solve the hit problem for the action of $\mathcal{A}$ on $\mathbf{P}(n)$, we might ask a prior question about the hit problem for $\mathbf{P}(n)$ as a $\mathcal{D}\otimes\mathbb{F}_2$-module, where we would expect fewer elements in a minimal generating set than in the Steenrod algebra case.

For two variables, the answer has been worked out by Walker and Xiao and appears in the second author's doctoral thesis.

Theorem 4.2 *For the action of $\mathcal{D}\otimes\mathbb{F}_2$ on $\mathbf{P}(2)$ a basis for $\mathbf{Q}(\mathbf{P}(2))$ is given by the monomials $1, x_1, x_2, x_1^2x_2, x_1^{2^n-1}x_2$ for $n \geq 1$.*

For comparison we quote the corresponding result for the Steenrod algebra.

Theorem 4.3 *For the action of $\mathcal{A}$ on $\mathbf{P}(2)$, a basis for $\mathbf{Q}(\mathbf{P}(2))$ is given by the the monomials $x_1^{2^k-1}x_2^{2^r-1}$ for $k, r \geq 0$, and $x_1^{2^a-1}x_2^{2^{a-1}-1+2^a(2^b-1)}$ for $a, b \geq 1$.*

In the general n-variable problem spikes are never hit under the action of the Steenrod algebra but can be hit under the action of differential operator algebra. There is a question about the exact relationship between the the two hit Problems 5.18.

4.2 The rational hit problem for $\mathcal{D}$

In this section we shall write $\mathbf{Q}(n)$ as a temporary notation for $\mathbf{Q}(\mathbf{W}(n)\otimes\mathbb{Q})$. It can be shown that $\mathcal{D}\otimes\mathbb{Q}$ is generated under composition by the operators D_k. In fact D_1, D_2 form a minimal algebraic generating set. The hit problem in this case reduces to the question of finding criteria on a polynomial g such that the differential equation

$$D_1f_1 + D_2f_2 = g$$

can be solved for polynomials f_1, f_2. In the two-variable case, it can be shown that $1, x_1, x_2, x_1x_2, x_1^2x_2$ form a basis of $\mathbf{Q}(2)$. In particular, the quotient is finite dimensional, as in Example 1.2. Furthermore, the differential equation $D_1f_1 + D_2f_2 = g$ can be solved for any homogeneous polynomial g of degree at least 4. Another similarity with Example 1.2 is that the monomials $x_1x_2, x_1^2x_2$ generate the regular representation of Σ_2 in $\mathbf{Q}(2)$. The monomial x_1x_2 generates the trivial representation, and the equation $D_1(x_1x_2) = x_1^2x_2 + x_1x_2^2$ shows that $x_1^2x_2$ generates the sign representation of Σ_2 in $\mathbf{Q}(2)$.

In the case of three variables, $n = 3$, it is shown in [43] that the monomials

$$1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3,$$

$$x_1^2x_2, x_1^2x_3, x_2^2x_3, x_1x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_2^2x_3^3$$

generate $\mathbf{Q}(3)$. The regular representation is generated by those monomials in the list which are divisible by $x_1x_2x_3$. This time, the differential equation $D_1f_1 + D_2f_2 = g$ can be solved if the homogeneous polynomial g has degree at least 7. In the general case of n variables, it is known that $\mathbf{Q}(n)$ is finite dimensional. However, the following conjecture, suggested by the above particular cases, seems harder to prove.

Conjecture 4.4 *For the action of the differential operator algebra $\mathcal{D} \otimes \mathbb{Q}$ on the polynomial algebra $\mathbb{Q}[x_1, \cdots, x_n]$, $\mathbf{Q}(n)$ contains the regular representation of the symmetric group Σ_n generated by the monomials divisible by the product of the variables $x_1 \cdots x_n$. In particular, the highest grading of $\mathbf{Q}(n)$ is $d = n(n+1)/2$ and, in this grading, $\mathbf{Q}^d(n)$ is the 1-dimensional sign representation of Σ_n, generated by the $x_1x_2^2 \cdots x_n^n$. Furthermore, monomials of the form*

$$x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n},$$

where $1 \leq i_r \leq r$, form a basis of the part of $\mathbf{Q}(n)$ divisible by $x_1 \cdots x_n$.

This conjecture implies, in particular, that every homogeneous polynomial f of degree greater than $n(n+1)/2$ is hit; in other words, the differential equation

$$D_1f_1 + D_2f_2 = g$$

can be solved for any g in these degrees. There is clearly a close connection between the representation theory of the symmetric group and the hit problem for the differential operator algebra. The decomposition of $\mathbf{W} \otimes \mathbb{Q}$ by a complete set of orthogonal idempotents associated with the irreducible representations of Σ_n is preserved by the action of $\mathcal{D} \otimes \mathbf{Q}$. The piece of $\mathbf{W} \otimes \mathbf{Q}$ corresponding to the trivial representation is the subspace of symmetric polynomials.

4.3 Remark

The algebra $\mathcal{D}$ preserves rings of invariants of permutation groups. More precisely, if $\Gamma \subset \Sigma_n$ is a subgroup the symmetric group, then $\mathbf{W}^\Gamma$ is a left module over $\mathcal{D}$. It would be interestind to investigate such modules both rationally and in the modular cases.

5 Problems

The following list of problems refers mainly to the prime 2 unless otherwise stated. There are of course analogous problems at any prime.

Problem 5.1 *Find a minimal generating set for* $\mathbf{P}(n)$ *as a module over* $\mathcal{A}$ *for* $n \geq 4$.

Problem 5.2 *Is the best bound for* $\dim \mathbf{Q}^d(\mathbf{P}(n))$ *the product* $(1)(3)\cdots(2^n-1)$*?*

Problem 5.3 *Is it true that the product of two non-hit polynomials in disjoint sets of variables is non-hit over* $\mathcal{A}$*?*

Problem 5.4 *Is it true that if a* monomial *in* $\mathbf{P}(n)$ *is non-hit over* $\mathcal{A}$*, then some matrix transformation of it contains a spike?*

Problem 5.5 *Find a formula for the excess of* $\widehat{\Theta}$*, where* Θ *is a composite of Steenrod squares, with a view to enhancing the use of the* χ*-trick.*

Problem 5.6 *Find a minimal generating set for* $\mathbf{B}(n)$ *as a module over* $\mathcal{A}$ *for* $n \geq 4$.

Problem 5.7 *If* f *is a hit monomial in* $\mathbf{P}(n)$ *over* $\mathcal{A}$*, is its symmetrisation* $\sigma(f)$ *hit symmetrically in* $\mathbf{B}(n)$*?*

Problem 5.8 *What is the best bound for the dimension of* $\mathbf{Q}^d(\mathbf{B}(n))$ *as a function of* n *independent of* d*?*

Problem 5.9 *Describe, for general* n*, the pieces of the maximal splitting of* $\mathbf{P}(n)$ *afforded by a complete set of orthogonal primitive idempotents in the semigroup ring* $\mathbb{F}_2[M(n, \mathbb{F}_2)]$*. How many distinct pieces are there? How many times does a piece occur. Find the Poincaré series of the pieces.*

Problem 5.10 *How do we write down the central idempotents in* $\mathbb{F}_2[M(n, \mathbb{F}_2)]$*? How do they decompose into primitives?*

Problem 5.11 *Describe the subalgebra of the semigroup algebra*

$$\mathbb{F}_2[M(n, \mathbb{F}_2]$$

whose action on $\mathbf{P}(n)$ *preserves symmetric functions. In particular find idempotents in this algebra.*

Problem 5.12 *Describe, for general* n*, the pieces of the maximal splitting of* $\mathbf{B}(n)$ *and* $\mathbf{M}(n)$ *afforded by a complete set of symmetry preserving orthogonal idempotents in the semigroup algebra* $\mathbb{F}_2[M(n, \mathbb{F}_2)]$*. How many pieces are there? How many times does a piece occur?*

Problem 5.13 *Solve the first occurrence problems for the irreducible modules of general linear groups as composition factors in the polynomial algebra at odd primes.*

Problem 5.14 *Solve the first occurrence problems for the irreducible modules of the symmetric groups in the polynomial algebra at odd primes.*

Problem 5.15 *Find the Poincaré series of the pieces* Y_λ *in the splitting of*

$$\Sigma(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty)$$

afforded by the irreducible representations λ *of the matrix semigroup* $M(n, \mathbb{F}_2)$.

Problem 5.16 *What is the maximal splitting of the stable homotopy type of* $BO(n)$*? How does it relate to the central idempotent splitting of* $\mathbf{B}(n)$*?.*

Problem 5.17 *Does the Thom complex* $MO(n)$ *split stably for* $n \geq 2$*?*

Problem 5.18 *What is the relation between the hit problems for* $\mathbf{P}(n)$ *as a module over* $\mathcal{A}$ *and as a module over* $\mathcal{D}$*?*

Problem 5.19 *Solve the hit problem for the action of* $\mathcal{D}$ *on* $\mathbf{W}(n)$ *for* $n \geq 2$ *over the rationals.*

Problem 5.20 *Investigate hit problems for the action of* $\mathcal{D}$ *on the pieces of* $\mathbf{W}(n)$ *split off by idempotents associated with irreducible representations of the symmetric group* Σ_n *in the modular case.*

Problem 5.21 *Is it true that the product of two non-hit polynomials in disjoint sets of variables is non-hit over* $\mathcal{D}$*?*

Problem 5.22 *What is the best bound for* $\dim(\mathbf{Q}^d(\mathbf{P}(n))$ *for* $\mathbf{P}(n)$ *as a module over* $\mathcal{D}$*?*

Problem 5.23 *Investigate rings of invariants of permutation groups as modules over* $\mathcal{D}$.

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